On the convergence of the method of alternating projections for multivariate symmetric eigenvalue problem

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Let Λ be a diagonal matrix of the form

$$\Lambda = diag\{\lambda_1 I^{[n_1]}, ..., \lambda_m I^{[n_m]}\}, \tag{1}$$

where $\sum_{i=1}^{m} n_i = n,\ \lambda_i \in R$ or $\lambda_i \in C$ for $i = 1, 2, ..., m$ and $I^{[n_i]}$ is the identity matrix of the size n_i. Let $x \in R^n$ be partotioned into blocks $x = [x_1^T, ..., x_m^T]^T$ with $x_i \in R^{n_i},\ i = 1, 2, ..., m.$

Definition. A multivariate eigenvalue problem (MEP) we define as: for a given symmetric matrix $A \in R^{n\times n}$ we should find the matrix Λ and vector x such that

$$Ax = \Lambda x,\ \|x_i\| = 1,\ i = 1, 2, ..., m, \tag{2}$$

where $\|x\|^2 = \sum_{i=1}^{k} x_i^2$ for $x \in R^k$.

Multivariare eigenvalue problems for symmetric and positive definite matrices appear in multivariate statistic theory where coefficients are to be determined so that the resulting linear combinations of sets of random variables are maximally correlated.

Algorithm. Let $x^0 \in R^n,\ \|x_i^0\| = 1,\ i = 1, 2, ..., m,$ be given. We define the method of alternative projections in the following way: let $k = 0$, here k denotes number of iteration. Given $x^k \in R^n$, solve the inhomogeneous symmetric eigenvalue problems for $i = 1, 2, ..., m$

$$A_{ii}x = \lambda_i x_i + b_i^k, \qquad \|x_i\| = 1 \tag{3}$$

where $b_i^k = -\sum_{j=1}^{i-1} A_{ij}x_j^{k+1} - \sum_{j=i+1}^{m} A_{ij}x_j^k$, and matrix A is partitioned into blocks, where $A_{ii} \in R^{n_i\times n_i},\ i = 1, 2, ..., m.$ Denote the solution of the problem as $(\lambda_i^{k+1}, x_i^{k+1})$. In the k-th iteration we get the diagonal matrix Λ^{k+1} and the vector $x^{k+1} \in R^n,\ k := k + 1.$

If we take the maximum (minimum) solutions of the inhomogeneous problems, then the method is convergent to the maximum (minimum) solution (Λ^*, x^*) of the multivariate symmetric eigenvalue problem. Here we prove the quadratic convergence of the method on the subspace

$$Q = Q_1 \times Q_2 \times ... \times Q_m \tag{4}$$

where $Q_i = \{y \in R^{n_i} : y = \alpha_i x_i^*,\ \alpha_i \in R\},\ i = 1, 2, ..., m.$

In generally, we prove only q-linear convergence of the method. In practice, for some examples the method is superlinearly convergent to the solution.