Iterative Refinement Techniques for Solving Block Linear Systems of Equations

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In many practical applications we need to solve a linear system of equations $Ax = b$, where $A \in \mathbf{R}^{N,N}$ is nonsingular and has special block structure. We assume that the matrix A is partitioned into $s \times s$ blocks, i.e. $A = (A_{ij})$, where $A_{i,j} \in \mathbf{R}^{n_i,n_j}$ is refereed to as the (i,j) block of A, $\{n_1, \dots, n_s\}$ is a given set of positive integers, $n_1 + \dots + n_s = N$.

Very often, the block matrices A_{ij} are sparse and many of them are zero. Numerical algorithms should exploit the structure of the matrix A. We would like to use algorithms that produce solutions y accurate to full machine precision. If $A + E$ has the same block structure as A: $A_{ij} = 0$ implies that $E_{ij} = 0$. If $A = (A_{ij})$ is symmetric then it is reasonable to have a numerical solution y being a solution of slightly perturbed symmetric system $(A + F)y = b$. We partly resolve this problem using blockwise approach. We prove that if $A = (A_{ij}) \in \mathbf{R}^{N,N}$ is a block symmetric matrix and y is a solution of a nearby linear system $(A + E)y = b$, then there exists $F = F^T$ such that y solves a nearby symmetric system $(A + F)y = b$, if A is symmetric positive definite or the matrix $\mu(A) = (\|A_{ij}\|_2)$ is diagonally dominant. Our blockwise analysis extend existing normwise and componentwise results on preserving symmetric perturbations.

We consider some ways in which iterative refinement may be used to improve the computed results. We present various kinds of iterative refinement techniques, eg. k-fold iterative refinement, for the solution of a nonsingular system $Ax = b$ with A partitioned into blocks using only single precision arithmetic. Fixed precision iterative refinement may give solutions to full single precision even when the initial solution have no correct significant figures. Very often, one or two steps are sufficient to terminate successfully the process. Extensive numerical testing was done in MATLAB to compare the performance of some direct methods for solving linear system of equations of special block matrices.

Some applications of the results for the least squares problem (LS) will be also considered.