Reduced averaging of directional derivatives in vertices of unstructured triangulations

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AMS subject classification: 65D25, 65N30

Key words: Conforming shape-regular triangulation, reduced averaging, superapproximation of the partial derivatives, recovery operator.

Let us consider a unit vector $z$, a conforming shape-regular triangulation $T_h$ without obtuse angles of a planar domain $\Omega$ with polygonal boundary and a smooth function $u = u(x, y)$ on $\Omega$. In many situations, the values of $u$ are known in the vertices of $T_h$ only. These values may be the results of some measurements or, more often, of numerical solutions of the second-order differential boundary-value problems. These known values can be naturally extended to the piecewise linear and globally continuous interpolant $\Pi_h(u)$ on $\Omega$. In some cases it is important to know accurate approximations of the values $\partial u/\partial z(a)$ in the vertices $a$ of $T_h$. As the approximations $\partial \Pi_h(u)/\partial z$, defined inside of the triangles from $T_h$, have an error of the size $O(h)$ only, the following classical problem appeared: For a unit vector $z$ and for the triangles $T_1, \ldots, T_n$ from $T_h$ with a common vertex $a$, find coefficients $f_1, \ldots, f_n$ such that the linear combination

$$f_1\partial(\Pi_h(u)|_{T_1})/\partial z + \ldots + f_n\partial(\Pi_h(u)|_{T_n})/\partial z$$

approximates $\partial u/\partial z(a)$ with an error $O(h^2)$.

In the case $n \geq 5$, we present an elementary construction of a selection $r = (b_1, \ldots, b_5)$ from the set of vertices of the triangles $T_1, \ldots, T_n$ different from $a$ such that the triangles $U_1 = ab_1b_5, \ldots, U_5 = ab_5b_1$ are equally oriented and $\angle(b_5ab_1) + \ldots + \angle(b_4ab_5) = 2\pi$ and denote by $\Pi_i(u)$ the linear interpolant of a function $u$ in the vertices of $U_i$ for $i = 1, \ldots, 4$. We derive a matrix $N(r)$ of size four, a vector $d$ and show that $N(r)$ is non-singular and that the value

$$\text{RA}_{h,z}[u](a) = g_1\partial\Pi_1(u)/\partial z + \ldots + g_4\partial\Pi_4(u)/\partial z$$

of the reduced averaging operator RA, related to the solution $g = [g_1, \ldots, g_4]^\top$ of the equations $N(r)g = d$, is a second-order approximation of $\partial u/\partial z(a)$ for any function $u \in C^3(\Omega)$. We show that this operator is more effective and more accurate than any other known operator approximating $\partial u/\partial z(a)$ locally. We discuss the problem of effective approximation of the values of the gradient $\nabla u$ in the vertices of $T_h$ and show that the piecewise linear extension of these approximations to $\Omega$ is a recovery operator in the sense of Ainsworth, Craig.