Reduced averaging of directional derivatives in vertices of unstructured triangulations

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Let us consider a unit vector z, a conforming shape-regular triangulation $\mathcal{T}_h$ without obtuse angles of a planar domain Ω with polygonal boundary and a smooth function $u = u(x, y)$ on Ω. In many situations, the values of u are known in the vertices of $\mathcal{T}_h$ only. These values may be the results of some measurements or, more often, of numerical solutions of the second-order differential boundary-value problems. These known values can be naturally extended to the piecewise linear and globally continuous interpolant $\Pi_h(u)$ of u on Ω. In some cases it is important to know accurate approximations of the values $\partial u/\partial z(a)$ in the vertices a of $\mathcal{T}_h$. As the approximations $\partial \Pi_h(u)/\partial z$, defined inside of the triangles from $\mathcal{T}_h$, have an error of the size $O(h)$ only, the following classical problem appeared: For a unit vector z and for the triangles $T_1, \ldots, T_n$ from $\mathcal{T}_h$ with a common vertex a, find coefficients $f_1, \ldots, f_n$ such that the linear combination

$$f_1\partial(\Pi_h(u)|_{T_1})/\partial z + \ldots + f_n\partial\left(\Pi_h(u)|_{T_n}\right)/\partial z$$

approximates $\partial u/\partial z(a)$ with an error $O(h^2)$.

In the case $n \geq 5$, we present an elementary construction of a selection $r = (b^1, \ldots, b^5)$ from the set of vertices of the triangles $T_1, \ldots, T_n$ different from a such that the triangles $U_1 = \overline{ab^5b^1}, \ldots$, $U_5 = \overline{ab^4b^5}$ are equally oriented and $\angle(b^5ab^1) + \ldots + \angle(b^4ab^5) = 2\pi$ and denote by $\Pi_i(u)$ the linear interpolant of a function u in the vertices of U_i for $i = 1, \ldots, 4$. We derive a matrix $N(r)$ of size four, a vector d and show that $N(r)$ is non-singular and that the value

$$\mathrm{RA}_{h,z}[u](a) = g_1\partial\Pi_1(u)/\partial z + \ldots + g_4\partial\Pi_4(u)/\partial z$$

of the *reduced averaging operator* RA, related to the solution $g = [g_1, \ldots, g_4]^\top$ of the equations $N(r)g = d$, is a second-order approximation of $\partial u/\partial z(a)$ for any function $u \in C^3(\overline{\Omega})$. We show that this operator is more effective and more accurate than any other known operator approximating $\partial u/\partial z(a)$ locally. We discuss the problem of effective approximation of the values of the gradient ∇u in the vertices of $\mathcal{T}_h$ and show that the piecewise linear extension of these approximations to Ω is a recovery operator in the sense of Ainsworth, Craig.