Global and linear rate of convergence of higher order methods on Powell's Singular Function

Trond Steihaug, and Sara Suleiman

Department of Informatics,

University of Bergen,

Norway

{Trond.Steihaug, Sara.Suleiman}@ii.uib.no

Key words: System of nonlinear equations, Halley's method.

To remove singularities at the root of a nonlinear equation $f(x) = 0$ Schröder in 1870 suggested to consider the nonlinear equation

$$g(x) \equiv f(x)/f'(x) = 0$$

provided that $f'(x) \neq 0$ in a neighborhood of the solution x^* except at x^*. We can extend this to a system of nonlinear equations $F(x) = 0$ where $F : \mathbb{R}^n \to \mathbb{R}^n$ is at least two times continuously differentiable. We now follow Schröder and apply Newton's method on $G(x) = 0$ where

$$G(x) \equiv F'(x)^{-1}F(x) = 0$$

provided the Jacobian matrix $F'(x)$ is nonsingular in a neighborhood of the solution x^* except possibly at x^*. The method will have second order rate of convergence under suitable assumptions. We show that Schröders method is closely related to the super-Halley method on $F(x) = 0$. Super Halley is a member in the Halley Class of methods. All methods in the Halley Class have third order rate of convergence under suitable assumptions. Other well known methods in the Halley class are the Chebyshev method and the (original) Halley method. We will explore relationship between these method and the underlying quadratic Taylor model these methods rely on. It will be shown that the methods inherit the rate of convergence from the approximation they generate of the quadratic model.

We will illustrate the methods using classical problems where the Hessian/Jacobian is singular at the solution and thus violates the assumption needed for higher rate of convergence. In particular we will consider the function introduced by M.J.D. Powell in 1962. The Powell singular function is an unconstrained optimization problem. However, the function is also used as nonlinear least squares problem and system of nonlinear equations. The function is a classic test function included in collections like MINPACK, Hoch and Schittkowski and CUTE as well as an example problem in text books. The function is convex and the Hessian or Jacobian matrix is singular at the solution. The function is stated as a difficult test case. In addition to the above methods we consider Newton's method and show that the methods have global convergence. However, they all have a linear rate of convergence. We will illustrate these properties with numerical experiments.