A method for the inverse numerical range problem

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The *numerical range* (also known as the *field of values*) of a square matrix $A \in \mathbf{C}^{n\times n}$ is the compact and convex set $F(A) = \{x^*A\,x \in \mathbf{C} : \, x \in \mathbf{C}^n,\, x^*x = 1\}$. The compactness follows readily from the fact that $F(A)$ is the image of the compact unit sphere of $\mathbf{C}^n$ under the continuous mapping $x \longmapsto x^*A\,x,$ and the convexity of $F(A)$ is the celebrated Hausdorff-Toeplitz Theorem. The concept of the numerical range and related notions has been studied extensively for many decades. It is quite useful in studying and understanding matrices and operators, and has applications in numerical analysis, differential equations, systems theory etc.

In this work, we propose a simple geometric algorithm for solving the inverse numerical range problem: *given an interior point* μ *of* $F(A)$, *determine a unit vector* $x_\mu \in \mathbf{C}^n$ *such that* $\mu = x_\mu^*A\,x_\mu$. The new algorithm is fast and gives numerically accurate results where known methods often fail, such as when μ lies in very close distance from the actual numerical range boundary $\partial F(A)$, both if $\mu \in F(A)$ and $\mu \notin F(A)$.