Improved Transparent Boundary Conditions for Pricing American Options

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In this paper, we present a new method for the approximation of transparent boundary conditions, when solving the American option pricing problem in financial mathematics. Using the standard change of variables cited in [1], the free boundary value problem for pricing American options is equivalent to the following problem:

$$\begin{cases} \frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2}, \quad a < x < x_f(\tau), \quad 0 < \tau \leq \tau^*, \\ u\left(x, \tau\right) = g(x, 0), \quad a < x < x_f(0), \\ u\left(x_f(\tau), \tau\right) = g\left(x_f(\tau), \tau\right), \quad 0 < \tau \leq \tau^*, \\ e^{(\alpha-1)x_f(\tau)+\beta\tau}\left(\frac{\partial u\left(x_f(\tau),\tau\right)}{\partial x} + \alpha u\left(x_f(\tau), \tau\right)\right) = 1, \quad 0 < \tau \leq \tau^*, \\ u\left(x, \tau\right) \to 0 \ \text{ as } \ x \to -\infty. \end{cases} \tag{1}$$

where $g(x, \tau) = e^{-\alpha x - \beta\tau}\max(e^x - 1, 0)$ and the free boundary has changed to $x_f(\tau) = \ln(S_f(t)/E)$. Using this transformation, the free boundary value problem for an American call option which pays dividend and has an unbounded domain, is reduced to a problem in a bounded domain (see §3 in [1]). Hence, the last equation of (1) is changed to:

$$\left.\frac{\partial u}{\partial x}\right|_{x=a} = \frac{1}{\sqrt{\pi}}\int_0^\tau \frac{\partial u\left(a, \lambda\right)}{\partial \lambda}\frac{d\lambda}{\sqrt{\tau - \lambda}}. \tag{2}$$

We call this equation as the transparent boundary condition (TBC) [2]. In this paper, we have proposed some new improvements in Han and Wu's approach [1] via explicit form of transparent boundary condition by expanding it in a Taylor series:

$$\phi(\lambda) = \sum_{k=0}^{+\infty}\frac{(\lambda - \tau)^k\phi^{(k)}(\tau)}{k!}. \tag{3}$$

Obviously, $\phi(\tau) = u(a, \tau)$ is defined and differentiable on $[0, \tau^*]$ (for some τ^*) so our Taylor expansion is meaningful. We prove then that the transparent boundary condition (3) is equivalent to the following equation:

$$\frac{\partial u}{\partial x}\Big|_{x=a} = \frac{1}{\sqrt{\pi}}\sum_{k=0}^{+\infty}\frac{(-1)^{k+1}4^k}{k!(2k-1)}\phi^{(k)}(\tau)\cdot\tau^{k-\frac{1}{2}} \tag{4}$$

and by this we obtain:

$$\frac{u(a,\tau)}{\sqrt{\tau}} \to 0 \ \text{ when } \ \tau \to 0^{+}. \tag{5}$$

Based on this approximation, we calculate the price of the corresponding American call option with high precision. By using the third-kind Chebyshev polynomials in the context of implicit Crank-Nicolson and explicit Dufort-Frankel techniques for solving the Black-Scholes equation, we will arrive at a high order of accuracy in price approximations. We illustrate the efficiency of the proposed method on two examples and compare it with perviously published results from the literature.

References

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