Some error estimates for the lumped mass finite element method for a parabolic problem

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We consider the model initial–boundary value problem

$$u_t - \Delta u = 0, \quad \text{in } \Omega, \quad u = 0, \quad \text{on } \partial\Omega, \quad \text{for } t \geq 0, \quad u(0) = v, \quad \text{in } \Omega, \tag{1}$$

where Ω is a bounded convex polygonal domain in $\mathbb{R}^2$. We study the spatially semidiscrete lumped mass method for (1), where we seek an approximation $\bar{u}_h(t) \in S_h$ of $u(t)$, with $S_h = \{\chi \in \mathcal{C}(\Omega) : \chi|_\tau \text{ linear}, \forall\, \tau \in \mathcal{T}_h;\ \chi|_{\partial\Omega} = 0\}$, $\{\mathcal{T}_h\}_{0<h<1}$ a family of regular triangulations of Ω, with h denoting the maximum diameter of the triangles $\tau \in \mathcal{T}_h$. The lumped mass solution $\bar{u}_h(t)$ is obtained by

$$(\bar{u}_{h,t}, \chi)_h + (\nabla \bar{u}_h, \nabla \chi) = 0, \quad \forall \chi \in S_h, \quad \text{for } t \geq 0, \quad \text{with } \bar{u}_h(0) = v_h, \quad (v, w) = \int_\Omega vw\, dx,$$

where $(v, w)_h$ is a given quadrature approximation of (v, w) and $v_h \in S_h$ an approximation of v. Improving earlier results, our aim is to show that

$$\|\bar{u}_h(t) - u(t)\| \leq Ch^2 t^{-1+q/2} |v|_q, \quad \text{for } t > 0, \quad q = 0, 1, 2, \tag{2}$$

with $|v|_0 = \|v\| = (v, v)^{1/2}$ the norm in $L_2(\Omega)$, $|v|_1$ the norm in $H_0^1(\Omega)$ and $|v|_2 = \|\Delta v\|$. We show (2) for $q = 2$, and for $q = 1$ under an inverse assumption on S_h. However, for $q = 0$, we are only able to show (2) under an additional hypothesis on $\mathcal{T}_h$, which is satisfied for symmetric triangulations. If this hypothesis on $\mathcal{T}_h$ is not satisfied, we are only able to show the nonoptimal order error estimate,

$$\|\bar{u}_h(t) - u(t)\| \leq Cht^{-1/2} \|v\|, \quad \text{for } t > 0.$$

In addition, we give examples of nonsymmetric partitions in one space dimension, where this assumption is not valid. We also discuss the application to time discretization by the backward Euler and Crank-Nicolson methods.