Properties of the Sylvester Hadamard matrices and their applications

Marilena Mitrouli[a],

[a]Department of Mathematics, University of Athens

Panepistemiopolis 15784, Greece

mmitroul@math.uoa.gr

Key words: Sylvester Hadamard matrices, Sign property, D-optimal designs, Pivot patterns.

In 1867 Sylvester had noted that if one took a ± 1 matrix, S, of order t whose rows are mutually orthogonal then $\begin{bmatrix} \text{S} & \text{S} \\ \text{S} & \text{-S} \end{bmatrix}$ was an orthogonal ± 1 matrix of order $2t$. Matrices of this form are called Sylvester Hadamard matrices and are defined for powers of 2.

The first few Sylvester-Hadamard matrices of orders $2^p,\ p = 1, 2, 3$ are given below. For information, we give in an additional last column the number of times the sign changes as we proceed from the first to the last element across the row:

$$S_2 = \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 1 & - & 1 \end{array}\right],\ S_4 = \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 1 & - & 1 & - & 3 \\ 1 & 1 & - & - & 1 \\ 1 & - & - & 1 & 2 \end{array}\right],\ S_8 = \left[\begin{array}{cccccccc|c} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & - & 1 & - & 1 & - & 1 & - & 7 \\ 1 & 1 & - & - & 1 & 1 & - & - & 3 \\ 1 & - & - & 1 & 1 & - & - & 1 & 4 \\ 1 & 1 & 1 & 1 & - & - & - & - & 1 \\ 1 & - & 1 & - & - & 1 & - & 1 & 6 \\ 1 & 1 & - & - & - & - & 1 & 1 & 2 \\ 1 & - & - & 1 & - & 1 & 1 & - & 5 \end{array}\right]$$

In this paper we study properties of Sylvester Hadamard matrices that lead to interesting applications. More specifically,

1. The sign property

If a ± 1 matrix of order n, S_n, has all the sign changes 0, 1, ..., $n - 1$ then its equivalent with a Sylvester Hadamard matrix.

This property is well known to users of the Walsh functions but has not been emphasized in the mathematical literature. This has prompted us to mention it explicitly here.

2. D-optimal designs embedded in Sylvester Hadamard matrices

We will study which D-optimal designs (± 1 matrices with maximal determinant) of dimension m can be embedded in a Sylvester Hadamard matrix of dimension n.

3. Pivot patterns of Sylvester Hadamard matrices

We will study the pivot patterns of S_{16}, S_{32}, S_{64} and we will try to answer open questions concerning the appearing pivot values such as:

Question For a Hadamard matrix of order 16, the determinant of its lower right 4×4 principal submatrix can take the value 8 only if the matrix is in the Sylvester Hadamard equivalence class ?