# A proposal of GS-based preconditioning applicable to restarted GMRES($k$) method

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As known well, preconditioning is a technique to improve convergence rate of Krylov subspace methods. ILU(0)-decomposition preconditioning without extra fill-in is a major preconditioning for nonsymmetric linear system. ILU(0) preconditioning, however, pays high cost to build preconditioner matrix and needs extra forward and backward substitutions. In this paper, we propose GS(Gauss-Seidel)-based preconditioning without forward and backward substitutions. Furthermore, GS-based preconditioning pays very low cost to build preconditioner matrix. Through numerical experiments, effectiveness of GS-based preconditioning for restarted GMRES($k$) method will be demonstrated.

By Krylov subspace methods, we consider to solve a large non-singular linear system of equations,

$$A\boldsymbol{x} = \boldsymbol{b} \tag{1}$$

where $A$ is a given nonsymmetric coefficient $(n \times n)$-matrix, and $\boldsymbol{x}$, $\boldsymbol{b}$ are a solution vector and right-hand side vector of order $n$, respectively. Krylov subspace of order $m$ is spaned by

$$K_m(A;\boldsymbol{r}_0) \quad := \quad \mathrm{span}\{\boldsymbol{r}_0, A\boldsymbol{r}_0, \ldots, A^{m-1}\boldsymbol{r}_0\}, \tag{2}$$

where $\boldsymbol{r}_0(= \boldsymbol{b} - A\boldsymbol{x}_0)$ is an initial residual vector. As known well, Krylov subspace methods are effective iterative methods for solving large linear systems of equations, e.g., restarted GMRES($k$) method is often used. In addition, preconditioning is well known as a technique to improve convergence rate of Krylov subspace methods. In fact, I(Incomplete)LU(0)-decomposition preconditioning is often used for solving many realistic problems. ILU(0) preconditioning can improve convergence rate of Krylov subspace methods. On the other hand, computational cost of building of preconditioner matrix in ILU(0) preconditioning is very expensive, and it needs extra forward and backward substitution. In this article, we propose GS(Gauss-Seidel)-based preconditioning without extra forward and backward substitution.

We present performance of restarted GMRES($k$) method without preconditioning and with ILU(0) and GS-based preconditionings in Table 1. "TRR" means the true relative residual of the approximate solution $\boldsymbol{x}_{n+1}$. Matrix "xenon1" is derived from Florida sparse matrix collection.

From Table 1, we can see that GS-based preconditioned GMRES($k$) method outperforms compared with other preconditioned GMRES($k$) methods in view of convergence rate and robustness.

Table 1: Performance of restarted GMRES($k$) without preconditioning, and with ILU(0) and GS-based preconditionings.

| matrix | method | $k$ | $\omega$ | itr. | pre-t [sec.] | itr-t [sec.] | total-t [sec.] | $\log_{10}$ (TRR) | memory [Mb] | ratio |
|---|---|---|---|---|---|---|---|---|---|---|
| xenon1 | GMRES | 20 | - | 8053 | 0.00 | 30.28 | 30.28 | -8.00 | 23.3 | |
| | | 100 | - | 1934 | 0.01 | 15.54 | **15.55** | -8.00 | 53.1 | 1.00 |
| | | 500 | - | 1072 | 0.01 | 25.78 | 25.79 | -8.00 | 203.2 | |
| | | 1000 | - | 949 | 0.01 | 42.20 | 42.21 | -8.00 | 394.4 | |
| | | 5000 | - | 949 | 0.01 | 41.16 | 41.17 | -8.00 | 2060.8 | |
| | ILU(0)-GMRES | 20 | - | max | - | - | - | -2.32 | 37.8 | |
| | | 100 | - | max | - | - | - | -2.83 | 67.5 | |
| | | 500 | - | 4183 | 0.10 | 110.41 | 110.51 | -8.00 | 217.7 | |
| | | 1000 | - | 1450 | 0.11 | 60.85 | 60.96 | -8.00 | 408.8 | |
| | | 5000 | - | 1055 | 0.10 | 53.70 | **53.80** | -8.02 | 2075.2 | 3.83 |
| | GS-GMRES | 20 | 1.0 | 1841 | 0.01 | 7.03 | 7.04 | -8.00 | 24.1 | |
| | | 100 | 1.0 | 764 | 0.01 | 5.43 | **5.44** | -8.00 | 53.8 | 0.39 |
| | | 500 | 1.0 | 568 | 0.01 | 12.86 | 12.87 | -8.01 | 204.0 | |
| | | 1000 | 1.0 | 558 | 0.01 | 15.07 | 15.08 | -8.00 | 395.1 | |
| | | 5000 | 1.0 | 558 | 0.01 | 15.14 | 15.15 | -8.00 | 2061.5 | |