On the use of Product Integration in Fractional Differential Equations

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Fractional differential equations are used in several areas, such as probability theory, biology, economics, control theory and physics for modeling systems exhibiting anomalous dynamics, usually characterized by an ultraslow diffusion.

The topic of this work concerns with the numerical solution of linear FDEs in the form

$$\begin{cases} D^{\alpha}_{t_0} y(t) + \lambda y(t) = f(t) \\ y(t_0) = y_0, \end{cases} \tag{1}$$

where $\alpha \in \mathbb{R}$ is the fractional order, $\lambda \in \mathbb{R}$, $y(t) : [t_0, T] \to \mathbb{R}$ and the forcing term $f(t)$ is assumed sufficiently smooth. With $D^{\alpha}_{t_0}$ we denote the fractional derivative operator, with respect to the origin t_0, according to the Caputo's definition

$$D^{\alpha}_{t_0} y(t) \equiv \frac{1}{\Gamma(1-\alpha)} \int_{t_0}^{t} \frac{y'(s)}{(t-s)^{\alpha}} ds,$$

where $\Gamma(\cdot)$ is the Euler's gamma function.

We present some results on a class of competitive and highly accurate Product Integration rules derived from an equivalent formulation of (1) in terms of a Volterra integral equation with a generalized Mittag–Leffler function in the kernel.

By means of the error analysis we show that the proposed rules allow to overcome the order barrier of classical PI rules and we discuss the way in which rules of higher order can be developed.

Some aspects related to the computational complexity are furthermore discussed.

Finally, some numerical tests are presented in order to confirm the theoretical findings .