Spectral Equivalence between Toeplitz and Trigonometric Matrix Algebras Matrices.

Paris Vassalos[a], Dimitrios Noutsos[b]

[a]Department of Computer Science, Athens University of Economics and Business, Athens, Greece

[b]Department of Mathematics, University of Ioannina, Ioannina, Greece

pvassal@aueb.gr,dnoutsos@uoi.gr

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Spectrally equivalence between two sequences of matrices is a property that plays an important role in the numerical solution of linear systems since it can be used to construct efficient preconditioners for the preconditioned conjugate gradient method, and to form optimal multigrid schemes.

In this work, we prove the existence of matrices, $\tau_n(f)$, belonging to τ algebra that are spectrally equivalent with ill conditioned Toeplitz matrices $T_n(f)$. For that, we assume that the generating function f is real valued, nonnegative, continuous, with isolated roots of maximum order $\alpha \in \mathbb{R}^+$. Specifically, we prove that for $0 \leq \alpha \leq 2$ there exist a proper clustering of the eigenvalues of $\tau_n(f)^{-1}T_n(f)$ around unity. For $2 < \alpha < 4$, a weak clustering for the spectrum of the aforemention matrix is achieved, where the minimum eigenvalue is bounded from bellow, while a constant number, independent of n, of eigenvalues tend to infinity. The results are generalized to cover the more interesting, from theoretical and practical point of view, case of Block Toeplitz with Toeplitz Blocks (BTTB), matrices. Based on these theoretical statements we propose τ preconditioners that lead to superlinear convergence both in $1D$ and $2D$ case when the condition number of the Toeplitz matrix is $o(n^4)$. Finally, we show that the spectrally equivalence also holds between circulant matrices and ill-conditioned Toeplitz matrices. The main difference is that the continuous symbol which generates the Toeplitz matrix should have discrete roots of order less than 2. We perform many numerical experiments, whose results confirm the validity of theoretical analysis.