Galerkin and Runge–Kutta methods: 
Unified formulation and a posteriori error analysis

Georgios Akrivis\textsuperscript{a}, Charalambos Makridakis\textsuperscript{b}, 
and Ricardo H. Nochetto\textsuperscript{c} 

\textsuperscript{a}Department of Computer Science, University of Ioannina, 
451 10 Ioannina, Greece 
\textsuperscript{b}Department of Applied Mathematics, University of Crete, 
714 09 Heraklion, Crete, Greece 
\textsuperscript{c}Department of Mathematics, University of Maryland, 
College Park, MD 20742, USA 

akrivis@cs.uoi.gr, makr@tem.uoc.gr, rhn@math.umd.edu 

\textit{AMS subject classification:} 65M15, 65M50 

\textit{Key words:} Parabolic equations, single-step methods, a posteriori error analysis, superconvergence. 

We consider the time discretization of initial value problems for linear parabolic equations in an abstract Hilbert space setting,

\begin{equation}
\begin{aligned}
& u'(t) + Au(t) = f(t), & 0 < t < T, \\
& u(0) = u^0,
\end{aligned}
\end{equation}

by single-step schemes. 

We cast Galerkin and Runge–Kutta methods into a unified formulation; the approximate solutions $U$ are piecewise polynomials in partitions of $[0, T]$. The residual $R$ of $U$, i.e., the amount by which $U$ misses being exact solution, is in general of suboptimal order. Therefore, the straightforward approach leads to suboptimal a posteriori error estimates. 

Using suitable reconstructions $\hat{U}$ of $U$, we derive optimal order, residual based a posteriori error estimates. 

These methods yield, under appropriate compatibility conditions, nodal approximations of order higher than the global order. We derive analogous a posteriori nodal superconvergence estimates.